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COMMENT

On Hamiltonian formulations of magnetic field line equations

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Abstract. In this comment we show the equivalence between the two canonical formulations in any coordinate system of magnetic field line equations of Janaki and Ghosh, and of Cary and Littlejohn, through the expression of the magnetic field in terms of the vector potential.

In a recent paper Janaki and Ghosh (1987) presented a Hamiltonian formulation of magnetic line equations. Although extension to non-orthogonal coordinate systems was briefly touched upon, the Hamiltonian was expressed in implicit form only in an orthogonal curvilinear coordinate system, in terms of the physical components of the magnetic field $\boldsymbol{B}$. Cary and Littlejohn (1983) have treated non-canonical Hamiltonian mechanics applied to magnetic field line flow as a function of the magnetic potential vector $\boldsymbol{A}$, related as usual to the magnetic field

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{1}$$

In this comment we show the full equivalence of both Hamiltonian formulations. The two descriptions we present in any coordinate system (whether curvilinear, non-orthogonal or not), precludes any question of loss of generality.

The Cary and Littlejohn approach assumes the Morozov and Solov'ev (1966) variational principle

$$\delta \int A_i(u)\, \mathrm{d}u^i = 0 \tag{2}$$

which is parameter independent and allows us to choose any particular coordinate as our independent variable. In equation (2) A_i is the i-covariant component of vector $\boldsymbol{A}$ and u^i is the i coordinate. In this comment, as in the treatise of Spain (1956), the Einstein sum convention for repeated indices is assumed and other tensorial notations are generously used.

The three A_i components are defined up to a gauge transformation

$$A_i \to A_i + \frac{\partial \chi}{\partial u^i} \tag{3}$$

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which is used by Cary and Littlejohn, with no loss of generality, to impose the constraint

$$A_2 = 0. \tag{4}$$

The variational principle becomes

$$\delta \int (A_1\,\mathrm{d}u^1 + A_3\,\mathrm{d}u^3) = 0 \tag{5}$$

which is formally equivalent to the Hamiltonian variational principle (Goldstein 1980) producing the Hamilton equations of mechanics

$$\delta \int (p_1\,\mathrm{d}u^1 - H\,\mathrm{d}t) = 0 \tag{6}$$

when the following identifications are assumed:

$$\begin{aligned} p_1 &= A_1(u^1, u^2, t) \\ H &= -A_3(u^1, u^2, t) \\ t &= u^3 \end{aligned} \tag{7}$$

where H is the Hamiltonian as a function of the non-canonical variables u^1 and u^2; p_1 is the canonical momentum conjugated to u^1, and t is the independent variable taking the place of time in Hamilton equations.

Although some of these equations are expressed by Cary and Littlejohn in a different notation or by using cylindrical coordinates, the generality of the coordinates or the magnetic field is emphasised.

We will not pursue here how Cary and Littlejohn, working with non-conjugated variables in phase space, developed this idea by a perturbation theory in terms of Lie transforms, nor the complexity of the resulting non-integrable mechanics (Whittaker 1937).

Janaki and Ghosh have expounded their Hamiltonian in terms of the magnetic field $\boldsymbol{B}$ in a curvilinear orthogonal system of coordinates. When their argument is generalised to any coordinate system it results in

$$\begin{aligned} p &= \int g^{1/2} B^3\,\mathrm{d}u^2 \\ H &= \int g^{1/2} B^1\,\mathrm{d}u^2 \\ t &= u^3 \end{aligned} \tag{8}$$

where g is, as usual, the determinant of the metric tensor in coordinates u^i; and B^i are contravariant components of the magnetic field $\boldsymbol{B}$. Equations (8) reduce to that of Janaki and Ghosh for an orthogonal system of coordinates.

Equivalence is obtained between the Cary and Littlejohn Hamiltonian, our equation (7), and equation (8) of Janaki and Ghosh, when we demonstrate both are identical.

The covariant components of the potential vector may be expressed as:

$$\begin{aligned} A_1 &= \int g^{1/2} B^3\,\mathrm{d}u^2 \\ A_3 &= -\int g^{1/2} B^1\,\mathrm{d}u^2 \end{aligned} \tag{9}$$

with $A_2 = 0$ as in equation (4). To complete the verification we use equation (1) in tensorial notation

$$g^{1/2}B^i = \frac{\partial A_j}{\partial u^k} - \frac{\partial A_k}{\partial u^j} \tag{10}$$

where i, j, k is any even permutation of indices 1, 2, 3. Substitution of expressions (4) and (9) on the right-hand side of (10) clearly produces the expected B^1 and B^3 components. If use is made of the property

$$\nabla \cdot \boldsymbol{B} = 0 \tag{11}$$

written in tensorial notation as

$$\frac{\partial}{\partial u^2}(g^{1/2}B^2) = -\frac{\partial}{\partial u^1}(g^{1/2}B^1) - \frac{\partial}{\partial u^3}(g^{1/2}B^3) \tag{12}$$

the B^2 component from (9) and (10) is obtained. This completes the proof of the equivalence.

In commenting upon other papers we have limited ourselves to considerations of the magnetic lines for a given magnetic field; the physical application is one of many limited by the restriction (11) of zero divergence of the vector field. One could, for example, determine the streamlines of an incompressible fluid in a steady state in a parallel way.

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